

VORTEX RECONNECTION IN THE THREE DIMENSIONAL NAVIER-STOKES EQUATIONS

ALBERTO ENCISO, RENATO LUCA, AND DANIEL PERALTA-SALAS

ABSTRACT. We prove that the vortex structures of solutions to the 3D Navier-Stokes equations can change their topology without any loss of regularity. More precisely, we construct smooth high-frequency solutions to the Navier-Stokes equations where vortex lines and vortex tubes of arbitrarily complicated topologies are created and destroyed in arbitrarily small times. This instance of vortex reconnection is structurally stable and in perfect agreement with the existing computer simulations and experiments. We also provide a (non-structurally stable) scenario where the destruction of vortex structures is instantaneous.

1. INTRODUCTION

A fundamental feature of inviscid incompressible fluids in three dimensions is that the vorticity is transported along the fluid flow. More precisely, if $u(x, t)$ is the velocity field of a fluid satisfying the 3D Euler equations,

$$\partial_t u + (u \cdot \nabla)u = -\nabla P, \quad \operatorname{div} u = 0 \quad u(\cdot, 0) = u_0,$$

the vorticity $\omega := \operatorname{curl} u$ is known to evolve according to the transport equation

$$\partial_t \omega = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega.$$

This equation ensures that the vorticity at time t can be written in terms of the initial vorticity ω_0 as

$$\omega(\cdot, t) = \phi_{t*} \omega_0,$$

that is, as the push-forward of the initial vorticity along the time t flow generated by the velocity field. It then follows that, as long as the solution of the Euler equations does not blow up, there are no changes in the topology of the vortex structures of the fluid, such as vortex tubes or vortex lines. Recall that a vortex line at time t is an integral curve of the vorticity frozen at time t and a vortex tube is a toroidal surface (that is, a smooth embedded torus) arising as a union of vortex lines.

In presence of viscosity, the vorticity is no longer transported along the flow because the diffusion gives rise to a different phenomenon known as vortex reconnection. In short, one says that a vortex reconnection has occurred at time T if the vortex structures at time T and at time 0 are not homeomorphic, so there has been a change of topology. For example, a certain vortex tube can break and there can appear vortex tubes or vortex lines that are knotted or linked in a different way as the initial vorticity.

As discussed e.g. in [13, 2] and references therein, there is overwhelming numerical and physical evidence for vortex reconnection. Particularly relevant for our purposes are the recent experimental results presented in [14, 10], where the authors

study how vortex lines and tubes of different knotted topologies reconnect in actual fluids using cleverly designed hydrofoils. It is worth mentioning that the authors observe that these vortex reconnections can occur in very small times even for fluids with small viscosity. From the computational point of view, the reconnection of a trefoil-shaped vortex tube has been recently studied in detail by Kerr [11].

In contrast with the wealth of heuristic, numerical and experimental results on this subject, a mathematically rigorous scenario of vortex reconnection has never been constructed so far. As discussed in [16], this is probably due to the fact that with purely analytical methods it is difficult to analyze the time evolution of the Navier–Stokes equations to show that vortex reconnection actually takes place. In the more complex but similar case of magneto-hydrodynamics, magnetic reconnection (that is, the breaking and topological rearrangement of magnetic field lines) is known to occur and has deep physical implications.

Our objective in this paper is to fill this long-standing gap by providing a rigorous mechanism of vortex reconnection in viscous incompressible fluids. Once one comes up with the mechanism, it is not hard to see that it is actually quite flexible, so here we will strive to present the mechanism in the simplest, least technical situation. We provide a detailed discussion of the role that every element plays in the proof of this result in Section 6. In view of the aforementioned experimental and numerical results [14, 10, 11], we will be particularly interested in proving that vortex structures of any knot or link type can be spontaneously created or destroyed.

The context in which we carry out the analysis is the 3D Navier–Stokes equations,

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u = -\nabla P, \quad \operatorname{div} u = 0, \quad u(\cdot, 0) = u_0.$$

We will impose periodic boundary conditions, so the spatial variable will take values in the torus \mathbb{T}^3 with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Hereafter the viscosity ν will be a fixed positive constant.

We will next state two results on rigorous vortex reconnection. The first one says that there can be vortex reconnection at arbitrarily small times. More generally, we will construct a finite cascade of reconnections at any sequence of times

$$T_1 < T_2 < \cdots < T_n,$$

meaning that there is a smooth solution to the Navier–Stokes equations, which one can even assume to be global, such that it has some vortex structures at time T_k (for each odd integer k) that do not have the same topology as any of the vortex structures present at the times T_{k-1} or T_{k+1} . Furthermore, the scenario of reconnection that we present is *structurally stable*, by which we mean that:

- (i) The vortex reconnection phenomenon occurs (with vortex structures of the same topology) for any initial datum that is close enough in $C^{4,\alpha}(\mathbb{T}^3)$ to the initial velocity discussed in the theorem.
- (ii) The existence of non-homeomorphic vortex structures occurs not only between the times T_k and $T_{k\pm 1}$ with k odd, but also between any nonnegative times t_k and $t_{k\pm 1}$ for which $|T_k - t_k| + |T_{k\pm 1} - t_{k\pm 1}|$ is small enough.

Notice that this condition ensures that the vortex reconnection is experimentally observable. The result can be stated as follows, where, for simplicity, when the toroidal surface encloses a bounded domain (and this condition is non-trivial when

periodic boundary conditions are considered in the equations), with some abuse of notation we will also refer to this domain as a vortex tube.

Theorem 1.1. *Given any constants $0 =: T_0 < T_1 < \dots < T_n$ and $M > 0$, for each odd integer k in $[1, n]$ let us denote by \mathcal{S}_k any finite collection of closed curves and toroidal domains (with pairwise disjoint closures but possibly knotted and linked) that are contained in the unit ball of \mathbb{T}^3 . Then there is a global smooth solution $u : \mathbb{T}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ of the Navier-Stokes equations, with a high-frequency initial datum of norm $\|u_0\|_{L^2} = M$ and of zero mean, which, for each odd integer $k \in [1, n]$, exhibits at time T_k a set of vortex lines and vortex tubes diffeomorphic to \mathcal{S}_k that is not homeomorphic to any of the vortex structures of the fluid at time T_{k-1} or T_{k+1} . This scenario of vortex reconnection is structurally stable.*

Remark 1.2. A more visual description of the reconnection process is the following. For a suitably chosen smooth but highly oscillatory initial datum u_0 , all the vortex structures of the corresponding solution at times T_k with k even wind around a direction of the torus, while for times T_k with k odd the solution presents a set of vortex structures \mathcal{S}_k of arbitrarily complicated knot types that is contained in a small ball. Hence the vortex structures \mathcal{S}_k must have been created at some time between T_{k-1} and T_k and are destroyed between T_k and T_{k+1} . This phenomenon is still observable both if one introduces small perturbations of the initial datum and if one measures the vortex structures at slightly different times. The roles of even and odd times can obviously be exchanged.

Remark 1.3. It is possible to prescribe the Reynolds number of the initial datum instead of its L^2 norm. Details are given in Remark 4.5.

The second result shows that when one starts with an initial vorticity that is not structurally stable, vortex reconnection can take place instantaneously. The idea is that one can show that a vortex tube present at time 0 does not need to survive for positive times, as the vortex lines sitting on that vortex tube can rearrange instantaneously and change their topology:

Theorem 1.4. *Given any $M > 0$, there is a global C^∞ solution of the Navier-Stokes equations $u : \mathbb{T}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$, with initial datum of norm $\|u_0\|_{L^2} = M$ and of zero mean, which has a vortex tube at time 0 that breaks instantaneously.*

Remark 1.5. More visually, the proof of the theorem shows that at time 0 the space \mathbb{T}^3 is covered by vortex tubes that wind around two of the directions of the torus and all the vortex lines are periodic or quasi-periodic and tangent to these tori. In contrast, at any small enough positive time one (or any finite number) of the initial invariant tori is broken, and the rearrangement of the associated initial vortex lines gives rise to vortex lines that are not periodic or quasiperiodic and are not tangent to a vortex tube. The destruction of this invariant torus creates isolated periodic vortex lines with intersecting stable and unstable manifolds.

Let us give some heuristic ideas about the proof of these results. The first key observation is that, in order to prove these results, the real enemy is not only the fact that the Navier-Stokes equations are notoriously difficult to analyze, but rather the need to prove that a certain vortex structure originating at time T is not diffeomorphic to any of the structures initially present in the fluid. The difficulty here is that, as one needs to consider all diffeomorphisms (not just the

flow of the velocity field), the way the diffeomorphisms can transform the vortex structures is unpredictable. For example, the diffeomorphism could map a certain vortex line into a curve of length 10^{-80} , which one cannot hope to control in a computer-assisted proof. This difficulty is not merely a mathematical oddity but a fundamental problem, as it is well known that integral curves and invariant tori of complicated topology and arbitrarily small size can bifurcate from vector fields with an extremely simple structure.

The proof of Theorem 1.1 hinges on choosing an initial datum that is the sum of several smooth but highly oscillatory fields \mathcal{W}_k , that is

$$u_0 = M \mathcal{W}_0 + \delta_1 \mathcal{W}_1 + \cdots + \delta_n \mathcal{W}_n ,$$

and involves an interplay between the (very large) frequencies of the fields and their relative sizes that ensures that, at time T_k , the vortex structures of the fluid are somehow related to those of \mathcal{W}_k . Key to make this argument work is to find two families of vector fields, which can be conveniently chosen to be Beltrami fields, with arbitrarily large frequencies and such that in the first family one can find vortex structures diffeomorphic to those in \mathcal{S}_k (so this family is used to construct \mathcal{W}_k when k is odd) whereas in the second family all the vortex structures are non-contractible. An essential property of these families is that they are “robustly non-equivalent”, meaning that any (uniformly) small perturbation of a member of the first family is not topologically equivalent to a small perturbation of any member of the second family, and viceversa. This is proved using suitable estimates for Beltrami fields with sharp dependence on the frequency and KAM-theoretic ideas. It is worth mentioning that the frequencies we need to consider in the proof of Theorem 1.1 are much larger than $\nu^{-1/2}$, which explains why there is no hope of promoting this scenario of vortex reconnection to the vanishing viscosity limit.

For the proof of Theorem 1.4 we start with a well chosen initial condition such that at time 0 the torus \mathbb{T}^3 is covered by a configuration of vortex tubes that is not structurally stable. We then resort to Melnikov’s theory to show that the evolution given by the Navier–Stokes equations breaks some of these vortex tubes (given by resonant invariant tori of the vorticity) instantaneously. This scenario of vortex reconnection does not survive in the vanishing viscosity limit either (see Remark 5.1).

In both cases we carry out the computations for initial data of zero mean and arbitrary L^2 norm. The global existence of the solutions follows from a suitable stability theorem for the Navier–Stokes equation and the fact that our initial data are small perturbations of Beltrami fields of high frequency and arbitrarily large norm.

It is worth stressing that, although we have carefully chosen the initial data not to introduce any inessential technicalities in the proofs of these results, the underlying ideas are quite flexible and can be applied to more general initial data. For instance, Beltrami fields are extremely useful both to make the proof as simple as possible and to allow us to efficiently deal with vortex structures of any topology. However, the only part of the argument where we would not have been able to do without them (at the expense of losing generality and making the proof more involved) is to show the global existence of the solutions, which is not essential for vortex reconnection.

The paper is organized as follows. Building on previous work of two of the authors [6], in Section 2 we construct high-frequency Beltrami fields on the torus with structurally stable vortex structures of prescribed topology. A key new feature here is that we derive fine estimates for the norm of these fields in terms of their frequency. In Section 3 we establish a stability theorem for the Navier-Stokes equations with periodic boundary conditions which yields some of the estimates needed for Theorem 1.1. The proof of Theorem 1.1, which consists of three main steps, is presented in Section 4, where we also discuss a variant of this result in which we prescribe the Reynolds number of the initial datum instead of its L^2 norm. Section 5 is devoted to the proof of Theorem 1.4 on the instantaneous destruction of vortex tubes. The paper concludes with some remarks about the role that certain terms play in the proofs and minor generalizations, which we present in Section 6.

2. HIGH-FREQUENCY BELTRAMI FIELDS WITH VORTEX STRUCTURES OF COMPLEX TOPOLOGY

Our objective in this section is to provide a result on the existence of high-frequency Beltrami fields with vortex lines and tubes of prescribed topology, similar to the one proved in [6] for the torus and the 3-sphere. The technical advantage that this result offers over [6] is that here we will control both the norm of the Beltrami field and the quantitative stability bounds in terms of the frequency. This is crucial for the proof of Theorem 1.1.

Let us begin by recalling that a Beltrami field on \mathbb{T}^3 is an eigenfunction of the curl operator:

$$(2.1) \quad \operatorname{curl} W = N W.$$

We will restrict our attention to Beltrami fields of nonzero frequency N , which are necessarily divergence-free and have zero mean:

$$\int_{\mathbb{T}^3} W \, dx = 0.$$

It is easy to check that the spectrum of curl consists of the points of the form $N = \pm|k|$, where $k \in \mathbb{Z}^3$ is a 3-vector of integer components. The most general Beltrami field of frequency N is a vector-valued trigonometric polynomial of the form

$$W = \sum_{|k|=\pm N} \left(b_k \cos(k \cdot x) + \frac{b_k \times k}{N} \sin(k \cdot x) \right),$$

where $b_k \in \mathbb{R}^3$ are vectors orthogonal to k : $k \cdot b_k = 0$.

We recall that a set \mathcal{S} of vortex lines or vortex tubes is *structurally stable* if they are preserved under $C^{4,\alpha}(U)$ -small perturbations of the velocity field modulo a small diffeomorphism of \mathbb{T}^3 that is close to the identity in $C^\alpha(\mathbb{T}^3)$ (a brief discussion of the Hölder norms taken to define stability can be found in Section 6). Here U is any fixed open subset of \mathbb{T}^3 that contains \mathcal{S} . With some abuse of notation, we will often denote by a *tube* a toroidal domain in \mathbb{T}^3 (or its boundary, which is an embedded torus), and when we say that two tubes are disjoint (or that a tube is disjoint from a curve) we mean that the intersection with the closure of the associated domains is empty.

Theorem 2.1. *Let \mathcal{S} be a finite union of closed curves and tubes (with pairwise disjoint closures, but possibly knotted and linked) in \mathbb{T}^3 that is contained in the unit ball. Then for any large enough odd integer N there exists a Beltrami field W satisfying (2.1) and a diffeomorphism Φ of \mathbb{T}^3 such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of W . This set is contained in the ball of radius $1/N$ and structurally stable in the sense that any field W' satisfying*

$$(2.2) \quad \frac{1}{N} \|\operatorname{curl} W - \operatorname{curl} W'\|_{C^{3,\alpha}} < \eta$$

has a collection of vortex structures given by $\Phi'(\mathcal{S})$, where Φ' is a diffeomorphism and η is a small N -independent constant. Furthermore, the field W is bounded as

$$\frac{1}{CN} < \|W\|_{L^2} < \frac{C}{\sqrt{N}}$$

with C a constant independent of N .

Proof. It was proved in [5] that there is a small positive constant $\lambda \in (0, 1)$ and a Beltrami field w on \mathbb{R}^3 that satisfies

$$\operatorname{curl} w = \lambda w,$$

falls off at infinity as $|w(x)| \leq C/|x|$ and has a collection \mathcal{S}' of vortex lines and vortex tubes diffeomorphic to \mathcal{S} (understood now as a subset of the unit ball \mathbb{B}_1 in \mathbb{R}^3). The set \mathcal{S}' is structurally stable under $C^{4,\alpha}$ -small perturbations and one can assume that \mathcal{S}' is also contained in \mathbb{B}_1 .

In view of the sharp decay of w at infinity, Herglotz's theorem (see e.g. [9, Theorem 7.1.27]) ensures that w can be written as

$$w(x) = \int_{\mathbb{S}^2} f(\xi) e^{i\lambda x \cdot \xi} d\sigma(\xi),$$

where \mathbb{S}^2 is the unit sphere with its canonical measure $d\sigma$ and f is a complex-valued function in $L^2(\mathbb{S}^2)$. Notice that, as w is real-valued, one necessarily has that $f(\xi) = \overline{f(-\xi)}$, and moreover

$$(2.3) \quad i\xi \times f(\xi) - f(\xi) = 0$$

in $L^2(\mathbb{S}^2)$ because w is a Beltrami field.

By density we can take a function $g \in C^\infty(\mathbb{S}^2)$ with $\|f - g\|_{L^2(\mathbb{S}^2)} < \varepsilon$, thereby granting that the field

$$w_1(x) := \int_{\mathbb{S}^2} g(\xi) e^{i\lambda x \cdot \xi} d\sigma(\xi)$$

satisfies

$$\|w - w_1\|_{C^0(\mathbb{R}^3)} < C\varepsilon.$$

Without loss of generality, we can assume that g also satisfies the condition $g(\xi) = \overline{g(-\xi)}$, which ensures that w_1 is real-valued.

For each odd integer N , consider the set

$$\mathcal{X}_N := \{\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3 : \operatorname{height}(\xi) = N\},$$

where we recall that the *height* of a rational point $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ is the least common denomination of the irreducible fractions defined by the components of ξ . Notice that if the height of ξ is N , then $N\xi \in \mathbb{Z}^3$, but that the converse is not necessarily true.

It was proved in [3] that \mathcal{X}_N becomes uniformly distributed as $N \rightarrow \infty$ through the odd integers, and that the cardinality of this set satisfies

$$(2.4) \quad \frac{N}{C} < |\mathcal{X}_N| < CN^2.$$

As g is smooth, the uniform distribution property implies that, for any large enough odd integer N (depending on ε), the field

$$(2.5) \quad w_2(x) := \frac{1}{|\mathcal{X}_N|} \sum_{\xi \in \mathcal{X}_N} g(\xi) e^{i\lambda x \cdot \xi}$$

approximates w_1 in the ball of radius 2:

$$\|w_1 - w_2\|_{C^0(\mathbb{B}_2)} < \varepsilon.$$

Notice that $-\xi$ is in \mathcal{X}_N whenever $\xi \in \mathcal{X}_N$, which ensures that w_2 is real-valued by the symmetry of the function g . Since w , w_1 and w_2 satisfy the Helmholtz equation $\Delta w + \lambda^2 w = 0$, standard elliptic estimates then ensure that

$$(2.6) \quad \|w - w_2\|_{C^{6,\alpha}(\mathbb{B}_1)} < C\varepsilon.$$

Taking into account the expression (2.5) for the field w_2 , let us define a real-valued vector field on \mathbb{T}^3 as

$$\widetilde{W}(x) := \frac{1}{|\mathcal{X}_N|} \sum_{\xi \in \mathcal{X}_N} g(\xi) e^{ix \cdot (N\xi)}.$$

To see that the periodic boundary conditions are indeed satisfied, notice that $N\xi \in \mathbb{Z}^3$ by the definition of the set \mathcal{X}_N . Furthermore, since the uniform distribution of \mathcal{X}_N ensures that

$$\lim_{N \rightarrow \infty} |\mathcal{X}_N| \|\widetilde{W}\|_{L^2}^2 = \lim_{N \rightarrow \infty} \frac{(2\pi)^3}{|\mathcal{X}_N|} \sum_{\xi \in \mathcal{X}_N} |g(\xi)|^2 = (2\pi)^3 \int_{\mathbb{S}^2} |g(\xi)|^2 d\sigma(\xi),$$

the estimate (2.4) implies that

$$(2.7) \quad \frac{1}{CN} < \|\widetilde{W}\|_{L^2} < \frac{C}{\sqrt{N}}$$

It follows from the bound (2.6) that in the unit ball one has

$$(2.8) \quad \left\| \widetilde{W}\left(\frac{\lambda}{N} \cdot\right) - w \right\|_{C^{6,\alpha}(\mathbb{B}_1)} < C\varepsilon.$$

Let us now define the vector field on \mathbb{T}^3

$$W := \frac{\text{curl}(\text{curl} + N)}{2N^2} \widetilde{W},$$

which is easily shown to be a real Beltrami field on the torus with frequency N . Notice that W is then close to \widetilde{W} for large N ; indeed, since \mathcal{X}_N becomes uniformly

distributed, as $N \rightarrow \infty$ through odd integers one has

$$\begin{aligned} \lim_{N \rightarrow \infty} |\mathcal{X}_N| \|\widetilde{W} - W\|_{L^2}^2 &= \lim_{N \rightarrow \infty} \frac{(2\pi)^3}{|\mathcal{X}_N|} \sum_{\xi \in \mathcal{X}_N} \left| \frac{i\xi \times (i\xi \times g(\xi) + g(\xi))}{2} - g(\xi) \right|^2 \\ &= (2\pi)^3 \int_{\mathbb{S}^2} \left| \frac{i\xi \times (i\xi \times g(\xi) + g(\xi))}{2} - g(\xi) \right|^2 d\sigma(\xi) \\ &\leq (2\pi)^3 \int_{\mathbb{S}^2} \left| \frac{i\xi \times (i\xi \times f(\xi) + f(\xi))}{2} - f(\xi) \right|^2 d\sigma(\xi) + C\varepsilon^2 \\ &= C\varepsilon^2. \end{aligned}$$

Here we have used the identity (2.3) and the fact that $\|f - g\|_{L^2(\mathbb{S}^2)} < \varepsilon$. Notice that, by (2.7), the above estimate readily implies that

$$\frac{1}{CN} < \|W\|_{L^2} < \frac{C}{\sqrt{N}},$$

as well as the bound

$$\|W - \widetilde{W}\|_{L^2} < \frac{C\varepsilon}{\sqrt{N}},$$

where we have used (2.4). Notice that, by the definition of W and the bound (2.8),

$$(2.9) \quad \left\| W\left(\frac{\lambda}{N} \cdot\right) - w \right\|_{C^{4,\alpha}(\mathbb{B}_1)} \leq C \left\| \widetilde{W}\left(\frac{\lambda}{N} \cdot\right) - w \right\|_{C^{6,\alpha}(\mathbb{B}_1)} < C\varepsilon.$$

The first part of the theorem then follows easily from the structural stability of \mathcal{S}' .

To complete the proof of the theorem, for convenience let us introduce a variable y that takes values in \mathbb{B}_1 and set $x := \lambda y/N$, which will eventually be interpreted as the original variables on \mathbb{T}^3 . Subscripts x or y will denote that the various norms are computed with respect to that variable and the curl operator will always be defined using the variable x .

The structural stability of the set \mathcal{S}' of vortex lines and tubes of w implies that if

$$(2.10) \quad \left\| \frac{1}{N} \operatorname{curl} W' - w \right\|_{C_y^{3,\alpha}(\mathbb{B}_1)} < 2\eta,$$

with η a certain constant that does not depend on N , then $\operatorname{curl} W'$ has a set of vortex lines and tubes diffeomorphic to \mathcal{S} and contained in the region $y \in \mathbb{B}_1$. Taking the curl and small ε , a short computation shows that the inequality (2.9) implies that

$$\left\| \frac{1}{N} \operatorname{curl} W - w \right\|_{C_y^{3,\alpha}(\mathbb{B}_1)} \leq \frac{C}{\lambda} \|W - w\|_{C_y^{4,\alpha}(\mathbb{B}_1)}$$

can be taken smaller than η . Hence the inequality (2.10) will automatically hold provided that we have chosen ε small enough and

$$\frac{1}{N} \|\operatorname{curl} W - \operatorname{curl} W'\|_{C_y^{3,\alpha}(\mathbb{B}_1)} < \eta.$$

Using now that

$$\begin{aligned}
\frac{1}{N} \|\operatorname{curl} W - \operatorname{curl} W'\|_{C_y^{3,\alpha}(\mathbb{B}_1)} &= \frac{1}{N} \sum_{j=0}^3 \|\nabla_y^j (\operatorname{curl} W - \operatorname{curl} W')\|_{C_y^0(\mathbb{B}_1)} \\
&\quad + \frac{1}{N} [\nabla_y^3 (\operatorname{curl} W - \operatorname{curl} W')]_{\alpha,y,\mathbb{B}_1} \\
&= \sum_{j=0}^3 \lambda^j N^{-1-j} \|\nabla_x^j (\operatorname{curl} W - \operatorname{curl} W')\|_{C_x^0(\mathbb{B}_{\lambda/N})} \\
&\quad + \lambda^{3+\alpha} N^{-4-\alpha} [\nabla_x^3 (\operatorname{curl} W - \operatorname{curl} W')]_{\alpha,x,\mathbb{B}_{\lambda/N}} \\
&\leq \frac{1}{N} \|\operatorname{curl} W - \operatorname{curl} W'\|_{C_x^{3,\alpha}(\mathbb{B}_{\lambda/N})} \\
&\leq \frac{1}{N} \|\operatorname{curl} W - \operatorname{curl} W'\|_{C_x^{3,\alpha}(\mathbb{T}^3)}
\end{aligned}$$

with $0 < \lambda < 1$, where $[\cdot]_{\alpha,y,\mathbb{B}_1}$ denotes the Hölder seminorm of exponent α computed with respect to the variable y in the domain \mathbb{B}_1 , the theorem follows. \square

3. A STABILITY RESULT FOR THE NAVIER-STOKES EQUATIONS

In the proofs of Theorems 1.1 and 1.4 we will need to estimate a solution $u(x, t)$ to the Navier-Stokes equations on \mathbb{T}^3 whose initial datum u_0 is a small perturbation of a Beltrami field W with frequency N . Our goal in this section is to provide a stability result for the Navier-Stokes equations that is very well suited for this task.

We will state this stability result in terms of perturbations of a solution $w(x, t)$ to the Navier-Stokes equations that is in $L^2(\mathbb{R}^+, W^{r,\infty}(\mathbb{T}^3))$ with $r \geq 1$. Notice that, if W is a Beltrami field with frequency N , then the solution with initial datum $w(\cdot, 0) = W$ is

$$w(x, t) := e^{-\nu N^2 t} W(x),$$

so the result obviously applies when the initial datum is a Beltrami field.

As a further simplification, we will assume that the initial datum has zero mean:

$$\int_{\mathbb{T}^3} u_0 \, dx = 0.$$

This will always be the case if the initial datum is a linear combination of Beltrami fields. As the average velocity is a conserved quantity of the Navier-Stokes equations, it will then follow that

$$\int_{\mathbb{T}^3} u(x, t) \, dx = 0$$

for all t .

Let us introduce some notation that we will use in the rest of the paper. Since we will only need to deal with symmetric tensors, we shall denote by \otimes the *symmetric tensor product*, namely

$$(v \otimes w)_{ij} := \frac{1}{2}(v_i w_j + w_i v_j).$$

We will also use the shorthand notation

$$|\nabla^m w|^2 := \sum_{|\alpha|=m} |\partial^\alpha w|^2.$$

In particular, for a time-dependent vector field $w(x, t)$ one can write

$$\|w\|_{H^r}^2 := \sum_{m=0}^r \int_{\mathbb{T}^3} |\nabla^m w|^2 dx, \quad \|w\|_{L^2 W^{r,\infty}}^2 := \sum_{m=0}^r \int_0^\infty \|\nabla^m w(\cdot, t)\|_{L^\infty}^2 dt.$$

It is worth stressing that, while the existence of stability theorems for the Navier–Stokes equations is not surprising (see e.g. [17] for a similar result ensuring the stability of solutions on \mathbb{R}^3 that belong to the space $L^4(\mathbb{R}^+, H^1(\mathbb{R}^3))$), the specific dependence of our bounds on the various norms of the unperturbed solution is key in the proof of Theorem 1.1. In particular, it is essential to ensure that our bounds do not depend exponentially on higher norms of the unperturbed solution $w \in L^2 W^{r,\infty}$, but only on its $L^2 L^\infty$ norm. To state the theorem in a form that will be particularly useful later on, we find it convenient to recursively define the quantities $Q_r^w(t)$ associated with a vector field $w(x, t)$ as

$$(3.1) \quad Q_0^w(t) := 1 \quad Q_r^w(t) := 1 + \sum_{m=0}^{r-1} Q_m^w(t) \int_0^t \|\nabla^{r-m} w(\cdot, \tau)\|_{L^\infty}^2 d\tau.$$

Theorem 3.1. *Given some $r \geq 1$ and any $\sigma < 1$, let w be a global solution to the Navier–Stokes equations in $L^2(\mathbb{R}^+, W^{r,\infty}(\mathbb{T}^3))$, with initial datum $w_0 := w(\cdot, 0)$ of zero mean. Then there is a positive constant C , which does not depend on w , such that for any divergence-free initial datum u_0 with zero mean and*

$$(3.2) \quad \|u_0 - w_0\|_{H^r} < \frac{1}{C} Q_r^w(\infty)^{-\frac{1}{2}} e^{-C\|w\|_{L^2 L^\infty}^2},$$

the corresponding solution $u(x, t)$ to the Navier–Stokes equations is global and satisfies

$$\|u(\cdot, t) - w(\cdot, t)\|_{H^m} \leq C Q_m^w(t)^{\frac{1}{2}} e^{C\|w\|_{L^2 L^\infty}^2} \|u_0 - w_0\|_{H^m} e^{-\nu\sigma t}$$

for all $0 \leq m \leq r$ and $t > 0$, with a σ -dependent constant.

Proof. Denoting by P_w the pressure function of w , it is readily checked that u is a solution of the Navier–Stokes equations with data u_0 and pressure P if and only if the difference $v := u - w$ satisfies the equation

$$(3.3) \quad \partial_t v + \operatorname{div}(v \otimes v + 2v \otimes w) - \nu \Delta v = -\nabla P_v, \quad \operatorname{div} v = 0, \quad v(\cdot, 0) = v_0.$$

Here $P_v := P - P_w$ and $v_0 := u_0 - w_0$. It is standard that for any $v_0 \in H^r$ with $r \geq 1$ there exists a local in time solution $v \in L_{\text{loc}}^2([0, T], H^{r+1}(\mathbb{T}^3))$, which is a continuous function of time with values in $H^r(\mathbb{T}^3)$, provided that T is small enough. Our goal is to show that the solution is actually global, and this will follow provided that we show that one can control the H^r norm of v .

To this end we will consider the evolution in time of the energies

$$(3.4) \quad h_m(t) := \sum_{j=0}^m \int_{\mathbb{T}^3} |\nabla^j v(x, t)|^2 dx$$

with $m \leq r$ and show that these quantities do not blow up for any finite t (and, in particular, are bounded at time T). In the case $m = 0$, it is enough to multiply Equation (3.3) by v and integrate by parts to obtain

$$\begin{aligned} \frac{1}{2} \frac{dh_0}{dt} &= -\nu \int_{\mathbb{T}^3} |\nabla v|^2 dx - \int_{\mathbb{T}^3} v_i v_j \partial_j w_i dx \\ &= -\nu \int_{\mathbb{T}^3} |\nabla v|^2 dx + \int_{\mathbb{T}^3} w_i v_j \partial_j v_i dx \\ &\leq -\nu \int_{\mathbb{T}^3} |\nabla v|^2 dx + \|w\|_{L^\infty} \|v\|_{L^2} \|\nabla v\|_{L^2} \\ &\leq -\nu \sigma \|\nabla v\|_{L^2}^2 + C \|w\|_{L^\infty}^2 h_0 \\ &\leq (-\nu \sigma + C \|w\|_{L^\infty}^2) h_0, \end{aligned}$$

which yields

$$(3.5) \quad h_0(t) \leq \|v_0\|_{L^2}^2 e^{-2\nu\sigma t + C \int_0^t \|w(\cdot, \tau)\|_{L^\infty}^2 d\tau}.$$

Here σ is any fixed number in the interval $(0, 1)$ and we have used that, as the mean of v is a conserved quantity for Equation (3.3), $v(\cdot, t)$ has zero mean for all t , so the Poincaré inequality on the torus ensures

$$\|\nabla v\|_{L^2} \geq \|v\|_{L^2}.$$

The estimate for h_m with $m \geq 1$ can be proved by induction. In view of the estimate (3.5), let us make the induction hypothesis that for any $0 \leq m \leq r-1$ the function h_m is bounded as

$$(3.6) \quad h_m(t) \leq C Q_m^w(t) e^{C \int_0^t \|w(\cdot, \tau)\|_{L^\infty}^2 d\tau} \|v_0\|_{H^m}^2 e^{-2\nu\sigma t}.$$

It is now enough to prove the bound for h_r . For this, let us commute the equation for v with the spatial derivative ∂^α , with a multiindex of order $|\alpha| \leq r$, to find

$$\partial_t \partial^\alpha v - \nu \Delta \partial^\alpha v + \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left[((\partial^\beta v + \partial^\beta w) \cdot \nabla) \partial^{\alpha-\beta} v + (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} w \right] = -\nabla \partial^\alpha P_v,$$

where as is customary the condition $\beta \leq \alpha$ and the combinatorial numbers should be understood componentwise. Multiplying this equation by $\partial^\alpha v$, integrating by parts and using the Cauchy-Schwartz inequality and the fact that $|\alpha| \leq r$ one then infers

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\partial^\alpha v|^2 + \nu \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 &\leq \varepsilon \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 + C \int_{\mathbb{T}^3} |w|^2 |\partial^\alpha v|^2 \\ &\quad + C \sum_{m=0}^{r-1} \int_{\mathbb{T}^3} |\nabla^m v|^2 |\nabla^{r-m} v|^2 + C \sum_{m=1}^r \int_{\mathbb{T}^3} |\nabla^m w|^2 |\nabla^{r-m} v|^2 \end{aligned}$$

where ε is a small positive constant and the constant C depends on ε , ν and r .

Using now the fact that $v(\cdot, t)$ has zero mean for all t and the Gagliardo-Nirenberg inequality for zero-mean fields

$$\|\nabla^m v\|_{L^\infty} \leq C \|\nabla^{m+2} v\|_{L^2}^{1/2} \|\nabla^m v\|_{L^6}^{1/2} \leq C \|\nabla v\|_{H^{m+1}}^{1/2} \|v\|_{H^{m+1}}^{1/2}$$

for $0 \leq m \leq r-1$, we obtain, recalling the definition (3.4) of the energies h_m , that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\partial^\alpha v|^2 + (\nu - \varepsilon) \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 &\leq C \|\nabla v\|_{H^r} h_r^{3/2} + C \|w\|_{L^\infty}^2 h_r \\ &\quad + C \sum_{m=1}^r \|\nabla^m w\|_{L^\infty}^2 \|v\|_{H^{r-m}}^2 \\ &\leq \varepsilon \|\nabla v\|_{H^r}^2 + C h_r^3 + C \|w\|_{L^\infty}^2 h_r + C \sum_{m=1}^r \|\nabla^m w\|_{L^\infty}^2 h_{r-m}. \end{aligned}$$

These inequalities are true for all times up to the maximal time of existence of the solution. Summing them over all multiindices α such that $|\alpha| \leq r$ and using that

$$\|\nabla v\|_{H^r} \geq \|v\|_{H^r}$$

by the Poincaré inequality on the torus (again exploiting that $v(\cdot, t)$ has zero mean), one then infers

$$\begin{aligned} \frac{dh_r}{dt} &\leq -2(\nu - c\varepsilon) \|\nabla v\|_{H^r}^2 + C h_r^3 + C \|w\|_{L^\infty}^2 h_r + C \sum_{m=0}^{r-1} \|\nabla^{r-m} w\|_{L^\infty}^2 h_m \\ &\leq -2(\nu - c\varepsilon) h_r + C h_r^3 + C \|w\|_{L^\infty}^2 h_r + C \sum_{m=0}^{r-1} \|\nabla^{r-m} w\|_{L^\infty}^2 h_m, \end{aligned}$$

where the constant c depends on r but not on ε .

As long as

$$(3.7) \quad h_r^2(t) \leq \delta$$

for some small constant δ that depends on ε and ν one clearly has

$$-2(\nu - c\varepsilon) h_r(t) + C h_r^3(t) \leq -2\nu \sigma h_r(t),$$

Hence, recalling the induction hypothesis (3.6) we deduce that as long as (3.7) holds one has

$$\begin{aligned} \frac{dh_r}{dt} &\leq -2\nu \sigma h_r + C \|w\|_{L^\infty}^2 h_r + \\ &\quad + C e^{-2\nu \sigma t + C \int_0^t \|w(\cdot, \tau)\|_{L^\infty}^2 d\tau} \sum_{m=0}^{r-1} \|\nabla^{r-m} w\|_{L^\infty}^2 Q_m^w(t) h_m(0). \end{aligned}$$

Thus using the Grönwall inequality and recalling the definition given in (3.1) of $Q_r^w(t)$ (which implies, in particular, that these quantities are non-decreasing), we arrive at

$$\begin{aligned} h_r(t) &\leq C e^{-2\nu \sigma t + C \int_0^t \|w(\cdot, \tau)\|_{L^\infty}^2 d\tau} \left(1 + \sum_{m=0}^{r-1} Q_m^w(t) \int_0^t \|\nabla^{r-m} w(\cdot, \tau)\|_{L^\infty}^2 d\tau \right) h_r(0) \\ &= C e^{-2\nu \sigma t + C \int_0^t \|w(\cdot, \tau)\|_{L^\infty}^2 d\tau} Q_r^w(t) h_r(0). \end{aligned}$$

In particular, the smallness assumption (3.7) is satisfied provided that the inequality (3.2) is satisfied for some large enough, ε -dependent constant C . Hence the solution exists for all positive times and the theorem follows from the bound for $h_r(t)$. \square

We shall next state as a corollary a concrete instance of the theorem that is precisely what we will need to apply later on. For the ease of notation, here and in what follows let us agree to say that two quantities q and q' satisfy the condition

$$q \ll q'$$

if $q < \frac{1}{C}q'$ for some large but fixed constant that does not depend on any relevant parameters. With this notation, the application of Theorem 3.1 that we will actually use is as follows:

Corollary 3.2. *Given $r \geq 1$ and any $\sigma \in [0, 1)$, let $w(x, t)$ satisfy the hypotheses of Theorem 3.1 with*

$$\|w\|_{L^2 W^{m, \infty}} < C(1 + N^{m-1}),$$

where N is a large constant, $0 \leq m \leq r$ and C is a constant that does not depend on N . Then if

$$\|u_0 - w_0\|_{H^r} \ll N^{1-r},$$

then the solution to the Navier-Stokes equations with initial datum u_0 is globally defined and satisfies, for all $0 \leq m \leq r$,

$$\|u(\cdot, t) - w(\cdot, t)\|_{H^m} \leq C(N^{m-1} + 1)e^{-\nu\sigma t}\|u_0 - w_0\|_{H^m}.$$

Proof. It suffices to note that $Q_m^w(t)$ is a non-decreasing function of t and that the assumption on $\|w\|_{L^2 W^{m, \infty}}$ implies that

$$Q_m^w(\infty) < C(1 + N^{2m-2})$$

for all $0 \leq m \leq r$. □

4. RECONNECTION OF VORTEX TUBES

Our goal in this section is to prove Theorem 1.1, thereby establishing that one can choose a smooth initial datum of arbitrary L^2 norm such that the corresponding solution to the Navier-Stokes equations features n vortex reconnection processes at arbitrarily small times $T_1 < \dots < T_n$. As we will see, the vortex structures that are created and destroyed can have arbitrarily complicated topologies.

For the sake of clarity, we will divide the proof in three steps, where we will construct a collection of Beltrami fields of arbitrarily high frequencies that are stably non-equivalent (Step 1), derive uniform estimates for the linearization of the Navier-Stokes equations around a high-frequency Beltrami field (Step 2), and make a clever choice of some free constants that will allow us to derive the desired result (Step 3). As we will discuss in Section 6, the mechanism of vortex reconnection that we have presented here is quite flexible. What is key, however, is to choose carefully the frequencies and relative sizes of the various terms that we will use to construct the initial datum: in a way, the heart of the matter is a delicate interplay between several vector fields, all of which are of high frequency, combined with a uniform topological non-equivalence result for certain of these fields.

Step 1: Beltrami fields that are stably topologically non-equivalent. In the proof of the theorem we will need to consider two different families of high-frequency Beltrami fields. The first family of Beltrami fields, which is not explicit but features robust contractible vortex structures of complicated topology, is obtained by repeatedly using Theorem 2.1:

Lemma 4.1. *For every odd integer $1 \leq k \leq n$, let \mathcal{S}_k be any finite collection of (pairwise disjoint but possibly knotted and linked) closed curves and tubes that is contained in the unit ball. Then for any large enough odd integers N_k there are Beltrami fields W_k on \mathbb{T}^3 with the following properties:*

- (i) $\operatorname{curl} W_k = N_k W_k$.
- (ii) W_k has a collection of vortex lines and vortex tubes that is diffeomorphic to \mathcal{S}_k , structurally stable (in the sense of (2.2)) and contained in the ball \mathbb{B}_{1/N_k} of radius $1/N_k$.
- (iii) For any nonnegative integer m , the H^m norm of W_k satisfies

$$\frac{N_k^{m-1}}{C} < \|W_k\|_{H^m} < C N_k^{m-\frac{1}{2}}$$

with a constant C that depends on m but not on N_k .

Proof. This immediately follows by applying Theorem 2.1 to the sets \mathcal{S}_k . Although in this theorem we had only stated L^2 bounds, the H^m bound is immediate because for a Beltrami field the norm $\|W_k\|_{H^m}$ is obviously equivalent to $N_k^m \|W_k\|_{L^2}$. \square

The second family of Beltrami fields that we need to consider is given by the fields

$$B_N := (2\pi)^{-3/2} (\sin Nx_3, \cos Nx_3, 0),$$

where N is a positive integer. Notice that B_N satisfies the equation

$$\operatorname{curl} B_N = N B_N$$

and has been normalized so that $\|B_N\|_{L^2} = 1$.

It is not hard to see that, for any integer N and any odd integer $1 \leq k \leq n$, the Beltrami fields W_k and B_N are not topologically equivalent, meaning that there are vortex structures in W_k that are not homeomorphic to any of the vortex structures of B_N . The idea here is that all the vortex structures of B_N are non-contractible, while W_k has a set of vortex structures diffeomorphic to \mathcal{S}_k , which is, in particular, contractible.

The central result of Step 1 is to show that this situation is robust, meaning that the same property is true for any suitably small perturbations of these fields. While this can seem pretty obvious at first sight, the proof is not trivial, as it employs in a key way that the integral curves of a uniformly (with respect to N) small perturbation of B_N are confined in narrow regions of \mathbb{T}^3 for all times. Without this bound, for very large times the perturbation could get the perturbed integral curve far from the region where the integral curve of B_N lies and this could translate into the perturbed field actually having contractible integral curves diffeomorphic to \mathcal{S}_k (for example, the integral curves might wind around a direction for a long time but then unwind until eventually becoming a complicated closed contractible curve). The key bound that prevents this from happening is obtained from a

KAM-type argument applied to a zero-mean divergence-free field that lives on a three-dimensional space.

We shall next state the “robust non-equivalence” result that we will need to prove the existence of vortex reconnection. To simplify the statements, with a slight abuse of notation, by a vortex line and a vortex tube of a general divergence-free field we will respectively mean an integral curve and a (domain bounded by an) invariant torus of its curl. We also recall that a vortex line or a vortex tube in \mathbb{T}^3 is *contractible* if and only if it is homeomorphic to a curve or tube contained in the unit ball.

Lemma 4.2. *For any positive integer N and any odd integer $1 \leq k \leq n$, suppose that W' and B' are any vector fields on \mathbb{T}^3 with*

$$\|W_k - W'\|_{H^r} + \|B_N - B'\|_{H^r} \ll 1$$

for some $r \geq 7$, where the implicit constant in this inequality does not depend on N or N_k . Then:

- (i) W' has a collection of vortex lines and vortex tubes diffeomorphic to \mathcal{S}_k .
- (ii) B' does not have any contractible vortex lines or vortex tubes.

Proof. In fact, and using that $\text{curl } B_N = N B_N$ and $\text{curl } W_k = N_k W_k$, we will prove the result under the weaker hypothesis that

$$\left\| W_k - \frac{1}{N_k} \text{curl } W' \right\|_{C^{3,\alpha}} + \left\| B_N - \frac{1}{N} \text{curl } B' \right\|_{C^{3,\alpha}} \ll 1.$$

The proofs of the statements for B' and W' are logically independent. The case of W' can be essentially read off Theorem 2.1 (and Equation (2.2)) using that for $r \geq 7$ the H^r norm controls the $C^{4,\alpha}$ norm. Specifically, notice that

$$\begin{aligned} \left\| W_k - \frac{1}{N_k} \text{curl } W' \right\|_{C^{3,\alpha}} &= \frac{1}{N_k} \left\| \text{curl } W_k - \text{curl } W' \right\|_{C^{3,\alpha}} \\ &\leq \frac{C}{N_k} \|W_k - W'\|_{C^{4,\alpha}} \leq \frac{C}{N_k} \|W_k - W'\|_{H^r}. \end{aligned}$$

Since $N_k \geq 1$, the factor $1/N_k$ means that one can even take $\|W_k - W'\|_{H^r} < \eta N_k$, but we will not need this improvement.

Let us now focus on the statement for the field B' . Up to reparametrization, the integral curves of $\text{curl } B'$ coincide with those of

$$B'' := \frac{1}{N} \text{curl } B' = B_N + b,$$

with $b := \frac{1}{N} \text{curl } B' - B_N$, so it suffices to prove the result for B'' . The claim for B' hinges on a uniform KAM theoretic estimate. Essentially, what this KAM estimate gives us is that if $x(s)$ is an integral curve of B'' whose initial condition $x^0 = (x_1^0, x_2^0, x_3^0)$ is such that $N x_3^0 \in I_j^n(\frac{1}{10})$ and $\|b\|_{C^{3,\alpha}}$ is smaller than some N -independent constant, then $N x_3(s) \in I_j^n(\frac{1}{5})$ for all times s . Here we are using the notation $I_j^n(\delta)$ for the (non-pairwise disjoint) intervals in the circle $\mathbb{R}/2\pi N\mathbb{Z}$

defined as

$$\begin{aligned} I_1^n(\delta) &:= \left\{ z : z - 2\pi n \in \left(-\frac{\pi}{4} - \delta, \frac{\pi}{4} + \delta \right) \mod 2\pi N \right\}, \\ I_2^n(\delta) &:= \left\{ z : z - 2\pi n \in \left(\frac{\pi}{4} - \delta, \frac{3\pi}{4} + \delta \right) \mod 2\pi N \right\}, \\ I_3^n(\delta) &:= \left\{ z : z - 2\pi n \in \left(\frac{3\pi}{4} - \delta, \frac{5\pi}{4} + \delta \right) \mod 2\pi N \right\}, \\ I_4^n(\delta) &:= \left\{ z : z - 2\pi n \in \left(\frac{5\pi}{4} - \delta, \frac{7\pi}{4} + \delta \right) \mod 2\pi N \right\}, \end{aligned}$$

where $0 \leq n \leq N-1$ is an integer. Let us complete the proof of the lemma under the assumption that this result is true. The proof will be presented later.

Let $x(s)$ be an arbitrary (possibly non-periodic) integral curve of B'' . Since for all s the third coordinate is such that $Nx_3(s)$ lies in an interval $I_j^n(\frac{1}{5})$, it follows that the i^{th} component of the field B_N (where $i = 1$ if j is even and $i = 2$ if j is odd) evaluated on this integral curve satisfies

$$|B_{N,i}(x(s))| > c$$

for all s and some positive constant c that does not depend on N . This is simply because, on each of these intervals, either the sine or the cosine are bounded away from 0. Notice that obviously

$$|B_i''(x(s))| > c - \|b\|_{C^0} > \frac{c}{2}$$

provided that $\|b\|_{C^0}$ is small enough. It then follows that, for the integral curve that we are considering, $x_i(s)$ defines a map $\mathbb{R} \rightarrow \mathbb{T}$ whose derivative is bounded away from zero. Hence the map winds around the circle \mathbb{T} either in the positive or the negative direction for all times, so, in particular, it is not homotopic to a constant map $\mathbb{R} \rightarrow \mathbb{T}$. Hence the integral curve $x(s)$ must be homotopically nontrivial, so it cannot be contractible.

It only remains to prove the auxiliary technical result that an integral curve of B'' whose initial datum has Nx_3^0 in $I_j^n(\frac{1}{10})$, then $Nx_3(s) \in I_j^n(\frac{1}{5})$ for all times provided that $\|b\|_{C^{3,\alpha}}$ is small enough. This is easier to obtain in the rescaled variables $y := Nx$ and $\sigma := (2\pi)^{-3/2}Ns$. To this end, let us start by considering the integral curve $y(\sigma)$ of B_N with initial condition $y^0 = (y_1^0, y_2^0, y_3^0)$. Since the ODE reads as

$$\frac{dy_1}{d\sigma} = \sin y_3, \quad \frac{dy_2}{d\sigma} = \cos y_3, \quad \frac{dy_3}{d\sigma} = 0,$$

the integral curves of B_N are explicitly given by

$$y_1(\sigma) = y_1^0 + \sigma \sin y_3^0, \quad y_2(\sigma) = y_2^0 + \sigma \cos y_3^0, \quad y_3(\sigma) = y_3^0.$$

Notice that y_j takes values in $\mathbb{R}/2\pi N\mathbb{Z}$ and that the regions in \mathbb{T}^3 defined by $\{y : Ny_3 \in I_j^n(\frac{1}{5})\}$ are invariant under the flow of the ODE.

It is easy to see that the field B_N satisfies non-degeneracy KAM conditions in the sense that the slope of the integral curves on each torus $\{y : y_3 = y_3^0\}$ varies from torus to torus (that is, it is not a constant function of y_3^0). Indeed, this slope is given by

$$\frac{dy_1/d\sigma}{dy_2/d\sigma} = \tan y_3^0$$

when $y_3^0 \in I_j^n(\frac{1}{5})$ with j odd and by

$$\frac{dy_2/d\sigma}{dy_1/d\sigma} = \cot y_3^0$$

when j is even. Since the derivative of the tangent or cotangent never vanishes (and the function is everywhere defined on these intervals), the twist condition of the KAM theorem is automatically satisfied. Hence [12] any divergence-free field $B'' = B_N + b$ of zero mean on \mathbb{T}^3 with b small enough in $C^{3,\alpha}$ has the following property: for any $\beta \in \mathbb{T}$, the field B'' has an invariant torus given by

$$\Phi(\{x \in \mathbb{T}^3 : x_3 = \beta\}),$$

where Φ is a $C^{3,\alpha}$ diffeomorphism with $\|\Phi - \text{id}\|_{C^\alpha} < C\|b\|_{C^{3,\alpha}}^{1/2}$. Since the invariant tori have codimension 1, there can be no diffusion, which ensures that for any integral curve of B'' ,

$$|y_3(\sigma) - y_e^0| = N|x_3(s) - x_3^0| < C\|b\|_{C^{3,\alpha}}^{1/2}$$

for all times s . In particular, if the initial datum has Nx_3^0 in $I_j^n(\frac{1}{10})$, then $Nx_3(s) \in I_j^n(\frac{1}{5})$ for all times provided that $\|b\|_{C^{3,\alpha}} \ll 1$. The lemma then follows upon noticing that

$$\|b\|_{C^{3,\alpha}} = \frac{1}{N} \|\text{curl } B_N - \text{curl } B'\|_{C^{3,\alpha}} \leq \frac{1}{N} \|B_N - B'\|_{C^{4,\alpha}} \leq \frac{C}{N} \|B_N - B'\|_{H^r}$$

and that the field b is obviously divergence-free and of zero mean because it is the curl of another field. \square

Step 2: Estimates for the solution. We shall next use the families of Beltrami fields B_N, W_k to construct a suitable initial datum of norm M for the Navier-Stokes equations which exhibits vortex reconnection at times T_1, \dots, T_n .

For this, let us consider large nonnegative integers N_0, \dots, N_n with

$$1 \ll N_n \ll N_{n-1} \ll \dots \ll N_2 \ll \sqrt{N_1},$$

and small positive real numbers $\delta_1, \dots, \delta_n$ with $\delta_{j+1} \ll \delta_j$. Take the solution to the Navier-Stokes equations with initial condition

$$u_0 := M \mathcal{W}_0 + \sum_{j=1}^n \delta_j \mathcal{W}_j,$$

where we are using the notation

$$\mathcal{W}_j := \begin{cases} B_{N_j} & \text{if } j \text{ is even,} \\ W_j & \text{if } j \text{ is odd} \end{cases}$$

and the fields B_N and W_j were defined in Step 1.

Let us define the function

$$w(x, t) := M \mathcal{W}_0(x) e^{-\nu N_0^2 t},$$

which satisfies the Navier-Stokes equations on $\mathbb{T}^3 \times \mathbb{R}^+$ and the hypotheses of Corollary 3.2. By the assumptions on δ_j, N_j and the bounds for W_j stated in Lemma 4.1, the field

$$v_0 := \sum_{j=1}^n \delta_j \mathcal{W}_j$$

is bounded as

$$(4.1) \quad \|v_0\|_{H^m} < C\delta_1(N_1^{m-\frac{1}{2}} + 1).$$

Hence it follows from Corollary 3.2 that if

$$(4.2) \quad \delta_1 N_1^{r+\frac{1}{2}} \ll N_0^{-r}$$

then the solution u is globally defined and the difference

$$v(x, t) := u(x, t) - w(x, t)$$

is bounded as

$$(4.3) \quad \|v(\cdot, t)\|_{H^m} \leq C(N_0^{m-1} + 1)e^{-\nu\sigma t}\|v_0\|_{H^m},$$

for all $0 \leq m \leq r+1$, where the implicit constant in this inequality depends on M . Here and in what follows, r is a fixed integer, which we can assume to be larger than or equal to 7 as in Lemma 4.2.

We shall need more estimates for the difference v . To derive them, notice that v satisfies the equations

$$\partial_t v + \operatorname{div}(v \otimes v + 2v \otimes w) - \nu \Delta v = -\nabla P_v, \quad \operatorname{div} v = 0, \quad v(\cdot, 0) = v_0.$$

It is then standard that the field v can then be written, using Duhamel's formula, as

$$(4.4) \quad v(\cdot, t) = e^{\nu t \Delta} v_0 - 2 \operatorname{Lin}(\cdot, t) - \operatorname{Bil}(\cdot, t),$$

where

$$(4.5) \quad e^{\nu t \Delta} v_0 = \sum_{j=1}^n \delta_j \mathcal{W}_j e^{-\nu N_j^2 t},$$

and the linear and bilinear terms are

$$\begin{aligned} \operatorname{Lin}(\cdot, t) &:= \int_0^t e^{\nu(t-s)\Delta} \mathbb{P} \operatorname{div}(v(s) \otimes w(s)) ds \\ \operatorname{Bil}(\cdot, t) &:= \int_0^t e^{\nu(t-s)\Delta} \mathbb{P} \operatorname{div}(v(s) \otimes v(s)) ds. \end{aligned}$$

Here we are writing $v(t) \equiv v(\cdot, t)$ for the ease of notation and \mathbb{P} denotes the Leray projector onto divergence-free fields with zero mean, which is given by the Fourier multiplier

$$\widehat{\mathbb{P}V}(k) := \frac{|k|^2 I - k \otimes k}{|k|^2} \widehat{V}(k),$$

which is understood as zero on the zero mode.

Equation (4.4) shows that the difference $v(x, t)$ consists of terms that evolve according to the heat equation and of two terms whose evolution is more involved. To deal with these terms we will utilize the following simple bounds for the heat kernel:

Lemma 4.3. *Let f be a function (or vector field) on \mathbb{T}^3 with zero mean. For any integer $m \geq 0$ and $s > 0$ one has*

$$(4.6) \quad \|e^{s\Delta} f\|_{H^m} \leq C s^{-\frac{m}{2}} \|f\|_{L^2},$$

$$(4.7) \quad \|e^{s\Delta} f\|_{H^m} \leq e^{-s} \|f\|_{H^m}.$$

Proof. For the first bound we use that, as f has zero mean,

$$\|e^{s\Delta}f\|_{H^m}^2 = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|k|^{2m}}{e^{s|k|^2}} |\widehat{f}(k)|^2 \leq C s^{-m} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |\widehat{f}(k)|^2 = C s^{-m} \|f\|_{L^2}^2,$$

where we have used Parseval's identity and the fact that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \geq \frac{z^m}{m!}$ for any integer m and any $z > 0$. Likewise,

$$\|e^{s\Delta}f\|_{H^m}^2 = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|k|^{2m}}{e^{s|k|^2}} |\widehat{f}(k)|^2 \leq e^{-s} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |k|^{2m} |\widehat{f}(k)|^2 = e^{-s} \|f\|_{H^m}^2,$$

where we have used that $|k| \geq 1$ for all $k \in \mathbb{Z}^3 \setminus \{0\}$. \square

In the following lemma we apply these estimates to obtain bounds for the linear and bilinear terms in the Navier-Stokes evolution with the right dependence on the free parameters δ_k, N_k . Here and in what follows, all the implicit constants in expressions of the form $N_0 \gg 1$ and in the various bounds that we derive depend on the fixed parameters ν, M, T_k , but not on the free parameters δ_k, N_k .

Lemma 4.4. *For any $t \geq T_1$ and $N_0 \gg N_1^{\frac{r-1}{2}}$, the above linear and bilinear terms are bounded as*

$$\begin{aligned} \|\text{Lin}(\cdot, t)\|_{H^r} &\leq C \delta_1 N_0^{-2}, \\ \|\text{Bil}(\cdot, t)\|_{H^r} &\leq C \delta_1^2 N_0^{r+1} N_1^{r+2}. \end{aligned}$$

Proof. Using the bounds (4.1) and (4.3) for $\|v_0\|_{H^m}$ and $\|v\|_{H^m}$, a standard bilinear estimate and the Sobolev embedding, one can readily obtain

$$\begin{aligned} \|v(s) \otimes v(s)\|_{H^{r+1}} &\leq C \|v(s)\|_{L^\infty} \|v(s)\|_{H^{r+1}} \leq C \|v(s)\|_{H^2} \|v(s)\|_{H^{r+1}} \\ &\leq C \delta_1^2 N_0^{r+1} N_1^{r+2} e^{-2\nu\sigma s}, \\ \|v(s) \otimes w(s)\|_{L^2} &\leq C \|w(s)\|_{L^\infty} \|v(s)\|_{L^2} \leq C \delta_1 e^{-\nu N_0^2 s}, \\ \|v(s) \otimes w(s)\|_{H^{r+1}} &\leq C \|w(s)\|_{L^\infty} \|v(s)\|_{H^{r+1}} + C \|v(s)\|_{L^\infty} \|w(s)\|_{H^{r+1}} \\ &\leq C \|w(s)\|_{L^\infty} \|v(s)\|_{H^{r+1}} + C \|v(s)\|_{H^2} \|w(s)\|_{H^{r+1}} \\ &\leq C \delta_1 (N_0^r N_1^{r+\frac{1}{2}} + N_0^{r+2} N_1^{\frac{3}{2}}) e^{-\nu N_0^2 s} \\ &\leq C \delta_1 N_0^{r+2} N_1^{\frac{3}{2}} e^{-\nu N_0^2 s}, \end{aligned}$$

where in the last bound we are assuming that $N_0 \gg N_1^{\frac{r-1}{2}}$.

One can now use these bounds with the heat kernel estimates stated in Lemma 4.3 to estimate the linear term as

$$\begin{aligned}
\|\text{Lin}(\cdot, t)\|_{H^r} &\leq \int_0^t \|e^{\nu(t-s)\Delta} \mathbb{P} \operatorname{div}(v(s) \otimes w(s))\|_{H^r} ds \\
&\leq C \int_0^{t/2} \|e^{\nu(t-s)\Delta} \mathbb{P}(v(s) \otimes w(s))\|_{H^{r+1}} ds \\
&\quad + C \int_{t/2}^t \|e^{\nu(t-s)\Delta} \mathbb{P}(v(s) \otimes w(s))\|_{H^{r+1}} ds \\
&\leq C \int_0^{t/2} (t-s)^{-\frac{r+1}{2}} \|v(s) \otimes w(s)\|_{L^2} ds \\
&\quad + C \int_{t/2}^t e^{-\nu(t-s)} \|v(s) \otimes w(s)\|_{H^{r+1}} ds \\
&\leq C\delta_1 \int_0^{t/2} (t-s)^{-\frac{r+1}{2}} e^{-\nu N_0^2 s} ds \\
&\quad + C\delta_1 N_0^{r+2} N_1^{\frac{3}{2}} \int_{t/2}^t e^{-\nu(t-s)} e^{-\nu N_0^2 s} ds \\
&\leq C\delta_1 N_0^{-2} + C\delta_1 N_0^r N_1^{\frac{3}{2}} e^{-\nu N_0^2 t/2} \\
&< C\delta_1 N_0^{-2}
\end{aligned}$$

for $N_0 \gg 1$ and $t \geq T_1$. Likewise,

$$\begin{aligned}
\|\text{Bil}(\cdot, t)\|_{H^r} &\leq C \int_0^t \|e^{\nu(t-s)\Delta} \mathbb{P}(v(s) \otimes v(s))\|_{H^{r+1}} ds \\
&\leq C \int_0^t e^{-\nu(t-s)} \|v(s) \otimes v(s)\|_{H^{r+1}} ds \\
&\leq C\delta_1^2 N_0^{r+1} N_1^{r+2}.
\end{aligned}$$

□

Step 3: Choice of the constants and conclusion of the proof. Our goal now is to choose the constants δ_k, N_k so that the vortex structures of the fluid at time T_k are those of the field \mathcal{W}_k , modulo a small deformation that does not change their topology. In order to do that, let us take a look at the formula (4.4) for the field $v(\cdot, t)$.

The strategy is to ensure that the terms $\text{Lin}(\cdot, t)$ and $\text{Bil}(\cdot, t)$ (which are harder to analyze) do not significantly contribute to the solution for times up to T_n . This enables us to essentially restrict our attention to the simple term $e^{\nu t \Delta} v_0$, where we will choose the constants so that the leading term at time T_k is precisely $\delta_k e^{-\nu N_k^2 T_k} \mathcal{W}_k$. To make things precise and derive effective estimates we will need to rescale the field at each time T_k .

To implement this strategy, let us start with the time $T_0 = 0$. Since

$$\|v_0\|_{H^r} \leq C\delta_1 N_1^{r-\frac{1}{2}}$$

by the bound (4.1) and taking into account that $\mathcal{W}_0 = B_{N_0}$, it is clear that if

$$(4.8) \quad \delta_1 N_1^{r-\frac{1}{2}} \ll 1,$$

then Lemma 4.2 ensures that the field

$$\tilde{u}_0 := M^{-1}u_0 = \mathcal{W}_0 + M^{-1}v_0$$

does not have any contractible vortex lines or vortex tubes. As $M^{-1}u_0$ is simply a rescaling of the solution u at time 0, we infer that if the above condition is satisfied, then all the vortex lines and tubes of the fluid at time 0 are non-contractible. By Lemma 4.2, this property is structurally stable (and is therefore satisfied for all small enough times).

Let us next consider the behavior of the fluid at time T_k , with $1 \leq k \leq n$. Here it is convenient to rescale the velocity field by defining

$$\tilde{u}_k := \delta_k^{-1} e^{\nu N_k^2 T_k} u(\cdot, T_k).$$

It is clear that

$$\begin{aligned} \tilde{u}_k = \mathcal{W}_k + \frac{M}{\delta_k} e^{-\nu(N_0^2 - N_k^2)T_k} \mathcal{W}_0 + \sum_{1 \leq j \neq k \leq n} \frac{\delta_j}{\delta_k} e^{\nu(N_k^2 - N_j^2)T_k} \mathcal{W}_j \\ + \delta_k^{-1} e^{\nu N_k^2 T_k} (\text{Lin}(\cdot, T_k) + \text{Bil}(\cdot, T_k)). \end{aligned}$$

Our objective is to choose the constants δ_j, N_j so that, for all $1 \leq k \leq n$,

$$(4.9) \quad \frac{\delta_{k-1}}{\delta_k} e^{-\nu(N_{k-1}^2 - N_k^2)T_k} \ll N_{k-1}^{-r},$$

$$(4.10) \quad \frac{\delta_{k+1}}{\delta_k} e^{\nu(N_k^2 - N_{k+1}^2)T_k} \ll N_{k+1}^{-r},$$

where $\delta_0 := M$ and the second condition is of course absent for $k = n$. As $N_k \gg N_{k+1}$, $\delta_k \gg \delta_{k+1}$ and $T_k < T_{k+1}$, it is not hard to see that if these conditions are satisfied for all $1 \leq k \leq n$, then

$$(4.11) \quad \left\| \frac{M}{\delta_k} e^{-\nu(N_0^2 - N_k^2)T_k} \mathcal{W}_0 + \sum_{1 \leq j \neq k \leq n} \frac{\delta_j}{\delta_k} e^{\nu(N_k^2 - N_j^2)T_k} \mathcal{W}_j \right\|_{H^r} \ll 1.$$

We will also impose that

$$\delta_k^{-1} e^{\nu N_k^2 T_k} (\|\text{Lin}(\cdot, T_k)\|_{H^r} + \|\text{Bil}(\cdot, T_k)\|_{H^r}) \ll 1$$

for all $1 \leq k \leq n$. Since

$$\delta_k^{-1} e^{\nu N_k^2 T_k} \ll \delta_{k+1}^{-1} e^{\nu N_{k+1}^2 T_k} \ll \delta_{k+1}^{-1} e^{\nu N_{k+1}^2 T_{k+1}},$$

as a consequence of (4.9)-(4.10), it follows that it suffices to impose that

$$(4.12) \quad \delta_n^{-1} e^{\nu N_n^2 T_n} (\delta_1 N_0^{-2} + \delta_1^2 N_0^{r+1} N_1^{r+2}) \ll 1,$$

where we have used the bounds for Lin and Bil derived in Lemma 4.4.

If $1 \leq k \leq n$ is odd, it then follows from Lemma 4.2 that if the above conditions for the constants are satisfied, the field \tilde{u}_k (which is just a rescaling of $u(\cdot, T_k)$) has a collection of vortex tubes and vortex lines diffeomorphic to the set \mathcal{S}_k , and that this set is structurally stable. Likewise, if k is even, if the conditions are satisfied Lemma 4.2 ensures that all the vortex lines and vortex tubes of \tilde{u}_k (and therefore of $u(\cdot, T_k)$) are non-contractible, and that this property is structurally stable. This automatically yields Theorem 1.1.

Hence it only remains to show that one can indeed choose the constants N_k, δ_k so that the conditions (4.2), (4.8)–(4.10) and (4.12) are satisfied. To prove this, starting with N_n , let us take any large constants

$$1 \ll N_n \ll N_{n-1} \ll \cdots \ll N_2 \ll \sqrt{N_1},$$

and set, for $1 \leq k \leq n-1$,

$$\rho_k := e^{-\nu N_k^2 (T_k + T_{k+1})/2}.$$

In view of the first term in the inequality (4.12), let us take $N_0 \gg N_1$ so that

$$N_0^2 \gg N_1^{r-1} e^{\nu T_n N_n^2} \prod_{k=1}^{n-1} \rho_k^{-1}.$$

Notice that this implies that $N_0 \gg N_1^{\frac{r-1}{2}}$, as assumed in Lemma 4.4.

In view of the second term in (4.12), we now set

$$\delta_1 := c N_0^{-r-1} N_1^{-r-2} e^{-\nu T_n N_n^2} \prod_{k=1}^{n-1} \rho_k,$$

where $c \ll 1$ is a small positive constant that does not depend on the frequencies. A short computation now shows that all the above conditions are satisfied if one now recursively sets, for $1 \leq k \leq n-1$,

$$\delta_{k+1} := \delta_k \rho_k.$$

Notice that one then has $\prod_{j=1}^{k-1} \rho_k = \delta_k / \delta_1$.

Theorem 1.1 is then proved, up to the minor issue that the L^2 norm of u_0 in the above construction is not M but

$$\|u_0\|_{L^2} = M + O(\delta_1).$$

The final result then follows upon replacing the initial condition u_0 by $q u_0$, with $q := M / \|u_0\|_{L^2}$ satisfying $|q - 1| < C \delta_1$. It is apparent that this factor does not change anything in the above arguments, so one readily obtains the desired result.

Remark 4.5. One can prove the same result where, instead of imposing that the L^2 norm of the initial datum is an arbitrary constant M , one imposes that its Reynolds number is M . We recall that the Reynolds number is usually defined as

$$\mathcal{R}(u_0) := \frac{\|(u_0 \cdot \nabla) u_0\|_{L^2}}{\nu \|\Delta u_0\|_{L^2}}.$$

The proof is exactly as above, the only difference being that the initial condition that one must take is

$$u_0 := \nu M N_0 \tilde{B}_{N_0} + \sum_{k=1}^n \delta_k \mathcal{W}_k,$$

where the constants N_k, δ_k can be chosen as before and we are considering the family of Beltrami fields

$$\tilde{B}_N := \sqrt{5} (2 \sin N x_3, \sin N x_1 + 2 \cos N x_3, \cos N x_1),$$

which satisfy $\operatorname{curl} \tilde{B}_N = N \tilde{B}_N$. This is the right normalization constant for \tilde{B}_N in this context, as it ensures that

$$\frac{\|(\tilde{B}_N \cdot \nabla) \tilde{B}_N\|_{L^2}}{\|\Delta \tilde{B}_N\|_{L^2}} = \frac{1}{N},$$

which easily implies that $\mathcal{R}(u_0)$ is essentially M .

Two important observations are in order. The first one is that the factor N_0^γ with $\gamma = 1$ that we have put in front of \tilde{B}_{N_0} does not change anything in the argument, while for $\gamma > 1$ it would change it dramatically. The reason is that the $L^2 L^\infty$ norm of $N_0^\gamma \tilde{B}_{N_0} e^{-\nu N_0^2 t}$, which appears in an exponential in Theorem 3.1, is uniformly bounded with respect to the parameter N_0 precisely for $\gamma \leq 1$. The second observation is that we have replaced B_{N_0} by \tilde{B}_{N_0} because B_{N_0} satisfies the algebraic identity

$$(B_{N_0} \cdot \nabla) B_{N_0} = 0,$$

and this would have made the Reynolds number artificially small. It is worth mentioning that the family \tilde{B}_N enjoys the same property of robust non-contractibility established for the family B_N in Lemma 4.2, and that the proof follows the same lines (the algebra gets a little more awkward, though).

5. INSTANTANEOUS DESTRUCTION OF VORTEX TUBES

In this section we will prove Theorem 1.4. Given a C^∞ function $h : \mathbb{T}^2 \rightarrow \mathbb{R}$, let us consider as initial vorticity the field

$$\begin{aligned} \omega_0 := M \big(\sin(x_3 + \varepsilon h), \cos(x_3 + \varepsilon h), \\ -\varepsilon \partial_1 h \sin(x_3 + \varepsilon h) - \varepsilon \partial_2 h \cos(x_3 + \varepsilon h) \big), \end{aligned}$$

where M is a constant (which will be thought of as large), $h \equiv h(x_1, x_2)$ and ε is a small positive constant that will be specified later. Notice that ω_0 is divergence-free and of zero mean, and that the associated initial velocity can be written as

$$(5.1) \quad u_0 := M \big(\sin(x_3 + \varepsilon h), \cos(x_3 + \varepsilon h), 0 \big) + \varepsilon M \nabla \phi,$$

where ϕ is the only solution to the elliptic equation on \mathbb{T}^3

$$\Delta \phi = \partial_2 h \sin(x_3 + \varepsilon h) - \partial_1 h \cos(x_3 + \varepsilon h), \quad \int_{\mathbb{T}^3} \phi \, dx = 0.$$

Notice that u_0 is then divergence-free and of zero mean.

Observe that the difference between the initial velocity and the Beltrami field of unit frequency $W := M(\sin x_3, \cos x_3, 0)$ is obviously bounded as

$$\|u_0 - W\|_{H^k} < CM\varepsilon.$$

Since the solution to the Navier-Stokes equations with datum W is $w(x, t) = e^{-\nu t} W(x)$, Theorem 3.1 ensures that if $M\varepsilon$ is small enough there is a global smooth solution $u(x, t)$ to the Navier-Stokes equations with initial datum u_0 .

To study the structure of the vortex tubes at time 0, it is convenient to start by noticing that ω_0 is conjugated to the above Beltrami field W . Specifically, given the volume-preserving diffeomorphism of \mathbb{T}^3

$$(5.2) \quad \Phi(x) := (x_1, x_2, x_3 + \varepsilon h(x_1, x_2)),$$

it can be readily checked that ω_0 can be written as the pullback of W by Φ :

$$\omega_0 = \Phi^* W.$$

Since the surfaces $x_3 = \text{constant}$ are vortex tubes for the Beltrami field W and the vortex lines on these tubes are periodic or quasi-periodic depending on whether the number $\tan x_3$ is rational or irrational, it stems that the same picture is valid for ω_0 , so at time zero the torus \mathbb{T}^3 is covered by vortex tubes with periodic or quasi-periodic vortex lines.

We shall next use Melnikov's theory to show that some of these vortex tubes (the ones covered by periodic vortex lines, which are resonant invariant tori of the vorticity) can break down instantaneously. For this we will need to consider the evolution of the equation for the vortex lines. The vorticity formulation for the Navier–Stokes equations ensures that

$$(5.3) \quad \omega = \omega_0 + t (\nu \Delta \omega_0 + (\omega_0 \cdot \nabla) u_0 - (u_0 \cdot \nabla) \omega_0) + O(t^2).$$

To study the vortex lines we introduce coordinates on \mathbb{T}^3 associated with the diffeomorphism (5.2), which we denote by

$$X \equiv (X_1, X_2, X_3) := (x_1, x_2, x_3 + \varepsilon h(x_1, x_2)).$$

In terms of these coordinates, the ODE for the vortex lines,

$$\frac{dx}{d\tau} = \omega(x, t),$$

where $\omega(x, t)$ is given by (5.3) and τ is the parameter of the vortex lines, reads as

$$(5.4) \quad \frac{dX}{d\tau} = M (\sin X_3, \cos X_3, 0) + t F + O(t^2).$$

The components of the vector field F are obtained by changing variables in Eq. (5.3). After expanding in ε and performing a straightforward but tedious computation one obtains that the third component of F can be written as

$$(5.5) \quad \begin{aligned} F_3 = 2\nu M \varepsilon^2 & (\sin X_3 \partial_2 h \partial_{22} h + \sin X_3 \partial_1 h \partial_{12} h - \cos X_3 \partial_2 h \partial_{12} h - \cos X_3 \partial_1 h \partial_{11} h) \\ & + (\sin X_3 \partial_1 + \cos X_3 \partial_2) \psi + O(\varepsilon^3), \end{aligned}$$

where ψ is certain function on \mathbb{T}^3 of order $O(\varepsilon)$ whose (rather awkward) expression will not be needed. Here the partial derivatives should be interpreted as

$$\partial^\alpha f \equiv (\partial^\alpha f)(X_1, X_2, X_3 - \varepsilon h(X_1, X_2)),$$

that is, as the composition of the function $\partial_x^\alpha f(x)$ (the derivatives being taken with respect to the original variables x) with the diffeomorphism Φ^{-1} that passes from the coordinates X to x . Notice that the term $O(\varepsilon^3)$ is not uniformly bounded in M .

To apply Melnikov's theory we shall write the ODE for the vortex lines as a non-autonomous dynamical system on the plane. To this end we shall restrict our attention to a region covered by vortex tubes of the initial vorticity and where $\sin X_3$ is nonnegative, such as

$$(5.6) \quad X_3 \in \left(\frac{\pi}{4}, \frac{3\pi}{8} \right),$$

and to small times, which is not a serious drawback as we intend to prove the instantaneous destruction of invariant tori. We shall then define, in this region and for small enough t , a new parameter for the vortex lines as

$$ds := (M \sin X_3(\tau) + t F_1(X(\tau), t) + O(t^2)) d\tau ,$$

so that Equation (5.4) can be written as

$$(5.7a) \quad \frac{dX_1}{ds} = 1 ,$$

$$(5.7b) \quad \frac{dX_2}{ds} = \cot X_3 + O(t) ,$$

$$(5.7c) \quad \frac{dX_3}{ds} = \frac{t F_3}{M \sin X_3} + O(t^2) .$$

Since one can integrate the first equation to find that

$$X_1 = s + \xi ,$$

where ξ is a constant, it is now enough to analyze the nonautonomous planar system defined by the second and third components of the above system with X_1 replaced by $s + \xi$. At $t = 0$, the integral curves of this planar system are

$$(5.8) \quad X_2 = X_2^0 + s \cot X_3^0 , \quad X_3 = X_3^0 ,$$

so, since $(X_2, X_3) \in \mathbb{T}^2$, it turns out that the planar integral curve with initial condition (X_2^0, X_3^0) is periodic if and only if $\cot X_3^0$ is a rational number, its period being $2\pi q$ if $\cot X_3^0$ is given by the irreducible fraction p/q .

Let us consider a vortex tube of the initial vorticity given in the coordinates X by the equation

$$\cot X_3 = \frac{p}{q} ,$$

with p, q coprime integers and p/q in the interval $(\cot \frac{3\pi}{8}, 1)$, the latter restriction coming from the inclusion (5.6). In terms of the associated nonautonomous planar system, the corresponding Melnikov function is [7, Theorem 4.6.2]

$$\mathcal{M}(\xi) := \int_0^{2\pi q} \frac{\cot X_3 F_3}{M \sin X_3} \Big|_{(s+\xi, X_2^0 + \frac{p}{q}s, \arctan \frac{q}{p})} ds$$

It is standard that $\mathcal{M}(\xi)$ does not depend on X_2^0 , so we will take $X_2^0 := 0$. Using the expression (5.5) for F_3 , one immediately finds that

$$\begin{aligned} \mathcal{M}(\xi) = & \frac{2p\nu\varepsilon^2}{q} \int_0^{2\pi q} \left(\partial_2 h \partial_{22} h + \partial_1 h \partial_{12} h - \frac{p}{q} \partial_2 h \partial_{12} h - \frac{p}{q} \partial_1 h \partial_{11} h \right) \Big|_{(s+\xi, \frac{p}{q}s, \arctan \frac{q}{p})} ds \\ & + \frac{p}{qM} \int_0^{2\pi q} \left(\partial_1 + \frac{p}{q} \partial_2 \right) \psi \Big|_{(s+\xi, \frac{p}{q}s, \arctan \frac{q}{p})} ds + O(\varepsilon^3) . \end{aligned}$$

It is clear that the second integral vanishes by periodicity because

$$\int_0^{2\pi q} \left(\partial_1 + \frac{p}{q} \partial_2 \right) \psi \Big|_{(s+\xi, \frac{p}{q}s, \arctan \frac{q}{p})} ds = \int_0^{2\pi q} \frac{d}{ds} \left(\psi \Big|_{(s+\xi, \frac{p}{q}s, \arctan \frac{q}{p})} \right) ds = 0 .$$

Our goal now is to choose the function h so that the Melnikov function vanishes and all its zeros are simple. It is not hard to see that there are many ways to

accomplish this. For example, one can make the computations with the choice

$$(5.9) \quad h(x_1, x_2) := \cos(px_1 - qx_2)$$

to obtain

$$(5.10) \quad \mathcal{M}(\xi) = -\frac{2\pi\nu\varepsilon^2 p(p^2 + q^2)^2}{q} \sin(2p\xi) + O(\varepsilon^3),$$

Since the vorticity is divergence-free and the diffeomorphism Φ is volume preserving, it is clear that the flow of (5.4) preserves the measure $dX := dX_1 dX_2 dX_3$. In turn, this implies that the flow of the rescaled system (5.7) preserves the measure

$$\rho dX,$$

with

$$\rho := M \sin X_3 + t F_1(X, t) + O(t^2),$$

which is well defined in the region (5.6) for small enough t .

Let us now show that the vector field V defined by the right hand side of the rescaled system (5.7) satisfies two important additional technical conditions. Firstly, the 2-form defined by

$$\alpha := i_V(\rho dX_1 \wedge dX_2 \wedge dX_3)$$

where i_V denotes the inner product of the vector field V with a 3-form, is exact. In order to see this, notice that $\rho V = \Phi_*\omega$ is the expression of the vorticity in the coordinates X , so one has

$$\alpha = i_{\rho V}(dX_1 \wedge dX_2 \wedge dX_3) = \Phi_*(i_\omega(dx_1 \wedge dx_2 \wedge dx_3)).$$

Let us denote by β the 1-form dual to the velocity field u , defined in terms of the Euclidean metric as

$$\beta(v) := u \cdot v.$$

Using differential forms to characterize $\omega := \text{curl } u$, it is standard that

$$i_\omega(dx_1 \wedge dx_2 \wedge dx_3) = d\beta,$$

which shows that α is exact, as claimed:

$$\alpha = \Phi_*(d\beta) = d(\Phi_*\beta).$$

The second technical fact is that the vector field V satisfies a twist condition. More precisely, by Equation (5.8) the period of the function $X_2(s)$ is $2\pi/\cot X_3^0$, whose derivative with respect to X_3^0 does not vanish in the interval (5.6), so the period of the vortex lines in the variable s is different, in general, on distinct vortex tubes.

Since these technical conditions are satisfied, one can then apply a Melnikov-type theorem [7, Theorem 4.8.3] to the function (5.10). The function $\mathcal{M}(\xi)$ has exactly 4 zeros in the interval $\xi \in [0, 2\pi/p)$, all of which are nondegenerate (meaning that the derivative does not vanish at these points). The theorem then ensures that the invariant torus of equation $\cot X_3 = p/q$ breaks down for all small enough positive times, and that only 4 among the period $2\pi q$ integral curves on this invariant torus survive for small positive times; furthermore, two of them are elliptic and the other two are hyperbolic.

Since $\rho V = \Phi_*\omega$, the integral curves of the field V are diffeomorphic to those of the vorticity, so from the above statement about the breakdown of the invariant

tori of V it stems that one of the vortex tubes at initial time (corresponding to $\Phi^{-1}(\{X : \cot X_3 = p/q\})$) also breaks down instantaneously.

Theorem 1.4 is then proved. More precisely, the Melnikov theory [7, Theorem 4.8.3] guarantees that, as a consequence of the instantaneous breakdown of the invariant torus $\cot X_3 = p/q$, at any small enough positive times there appear integral curves (homoclinic or heteroclinic connections) that are not periodic or quasiperiodic and are not tangent to an invariant torus. The Melnikov theory also ensures that, by bifurcation, for any small positive time the destruction of this invariant torus gives rise to two elliptic periodic integral curves and to two hyperbolic periodic vortex lines with intersecting stable and unstable manifolds.

Remark 5.1. It follows from the proof of the theorem that, in fact, the contribution of the nonlinear term $(\omega_0 \cdot \nabla)u_0 - (u_0 \cdot \nabla)\omega_0$ to the Melnikov function is zero, as one can infer from the fact that the Euler equation does not feature vortex reconnection. This is a general fact, and does not depend on our choice of the initial datum. Notice, moreover, that Equation (5.10) ensures that the size of the perturbation, ε , must be of order $o(\nu)$, which explains why this scenario of instantaneous vortex reconnection does not survive in the vanishing viscosity limit.

Remark 5.2. A very minor modification of the argument permits to break instantaneously any finite number of the initial configuration of vortex tubes, not just one.

6. CONCLUSIONS

In this section we shall collect a number of remarks and observations about the proofs of Theorems 1.1 and 1.4 that provide further insight into this scenario of vortex reconnection.

Firstly, notice that the PDE aspects of the proofs of Theorems 1.1 and 1.4 are essentially linear, in the sense that we are concerned with small perturbations of a solution to the Navier-Stokes equations of the form $w(x, t) := M e^{-\nu N_0^2 t} B_{N_0}(x)$ with B_{N_0} a Beltrami field of high frequency N_0 . Notice that, as vortex reconnection is a dissipative effect, it should not be too surprising that the proof can be carried out in an essentially linear regime. This is seen very clearly in Remark 5.1. The way this should be interpreted is that, although the evolution of the fluid takes place, in general, in the nonlinear regime, in the creation or destruction of vortex structures the straw that actually breaks the laden camel's back is in fact essentially linear.

The heart of the proof is a high-frequency analysis in which we study which one among several terms of different frequencies (all of which are large) is dominant at different time scales $T_0 < T_1 < \dots < T_n$. Although this analysis is made simpler and finer by the fact that the terms can be taken as Beltrami fields, the underlying interplay between different frequencies could have been carried out in more general situations. Notice that, for any choice of the distinct-frequency terms, to rigorously pass from the frequency analysis to the statement that there is indeed the change of topology that constitutes the vortex reconnection, it is essential to have a “stable topological non-equivalence” theorem like our Lemma 4.2. This result can be extended to cover more general families of vector fields, but in all cases it is a quite non-trivial KAM-theoretic argument in itself.

The proof of the aforementioned Lemma 4.2 (and therefore that of Theorem 1.1) uses in a crucial way the periodic boundary conditions, that is, the fact that the spatial variable takes values in \mathbb{T}^3 . This is because this allows us to play with the quite robust concept of contractibility. In contrast, this condition does not play a role in the proof of Theorem 1.4, and in fact one can establish a similar result on \mathbb{R}^3 with a completely analogous reasoning.

It is also worth mentioning that the reconnection mechanism that we have presented in this paper does not depend much on the form of the dissipative term of the Navier–Stokes equations. Indeed, the fact that the dissipative term is given by the Laplacian has only been used to write that the time-dependent factor that appears in solutions to the Navier–Stokes equations whose initial datum is a Beltrami field goes as the exponential of the squared frequency, which is not essential, and to derive heat kernel estimates. In particular, the argument goes through for analogs of the Navier–Stokes equations that feature fractional dissipation of the form

$$\partial_t u + (u \cdot \nabla)u + \nu(-\Delta)^\alpha u = -\nabla P, \quad \operatorname{div} u = 0, \quad u(\cdot, 0) = u_0$$

with $\alpha > 0$.

The fact that unstable vortex structures such as resonant tori can break down instantaneously explains why there is abundant literature on “possibly robust” vortex structures, such as vortex lines and vortex tubes (which are robust when they satisfy suitable non-degeneracy conditions), but not on, for example, “vortex balls” or “vortex pretzels” (that is, spheres or genus-2 tori consisting of vortex lines), for which no robustness properties are expected in the theory of divergence-free dynamical systems.

To conclude, it is worth explaining why the difference between two fields needs to be controlled in several parts of the proofs through the norm $\|\operatorname{curl} W - \operatorname{curl} W'\|_{C^{3,\alpha}}$ (or, for convenience, through the stronger norms $\|W - W'\|_{C^{4,\alpha}}$ or $\|W - W'\|_{H^7}$). The reason is that controlling the vortex tubes of W' in terms of those of W essentially boils down to controlling invariant circles of suitable annulus diffeomorphisms defined by the flow of the fields $\operatorname{curl} W$ and $\operatorname{curl} W'$, and in this context it is known that $C^{3,\alpha}$ bounds are sufficient [8] (and essentially necessary [1]) for the convergence of a KAM scheme.

ACKNOWLEDGMENTS

The authors are supported by the ERC Starting Grants 633152 (A.E.), 277778 (R.L.) and 335079 (D.P.-S.). This work is supported in part by the ICMAT–Severo Ochoa grant SEV-2015-0554.

REFERENCES

1. C.Q. Cheng, L. Wang, Destruction of Lagrangian torus for positive definite Hamiltonian systems, *Geom. Funct. Anal.* 23 (2013) 848–866.
2. P. Constantin, Eulerian–Lagrangian formalism and vortex reconnection, in: *Mathematical aspects of nonlinear dispersive equations* (J. Bourgain, C.E. Kenig, S. Klainerman, Eds.), 157–170, Princeton University Press, Princeton, 2007.
3. W. Duke, Rational points on the sphere, *Ramanujan J.* 7 (2003) 235–239.
4. A. Enciso, D. Peralta-Salas, Knots and links in steady solutions of the Euler equation, *Ann. of Math.* 175 (2012) 345–367.

5. A. Enciso, D. Peralta-Salas, Existence of knotted vortex tubes in steady Euler flows, *Acta Math.* 214 (2015) 61–134.
6. A. Enciso, D. Peralta-Salas, F. Torres de Lizaur, Knotted structures in high-energy Beltrami fields on the torus and the sphere, *Ann. Sci. Éc. Norm. Sup.*, to appear (arXiv:1505.01605).
7. J. Guckenheimer, P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer-Verlag, New York, 1990.
8. M. Herman, Sur les courbes invariantes par les difféomorphismes de l’anneau, *Astérisque* 103-104, 1983, 221 pp.
9. L. Hörmander, *The analysis of linear partial differential operators I*, Springer, Berlin, 2003.
10. M.W. Scheeler, D. Kleckner, D. Proment, G.L. Kindlmann, W.T.M. Irvine, Helicity conservation by flow across scales in reconnecting vortex links and knots, *Proc. Nat. Acad. Sci.* 111 (2014) 15350–15355.
11. R.M. Kerr, Simulated Navier-Stokes trefoil reconnection, arXiv:1509.03142.
12. B. Khesin, S. Kuksin, D. Peralta-Salas, KAM theory and the 3D Euler equation, *Adv. Math.* 267 (2014) 498–522.
13. S. Kida, M. Takaoka, Vortex reconnection, *Annu. Rev. Fluid Mech.* 26 (1994) 169–189.
14. D. Kleckner, W.T.M. Irvine, Creation and dynamics of knotted vortices, *Nature Phys.* 9 (2013) 253–258.
15. A.J. Majda, A.L. Bertozzi, *Vorticity and incompressible flow*, Cambridge University Press, Cambridge, 2002.
16. K. Ohkitani, P. Constantin, Numerical study of the Eulerian–Lagrangian formulation of the Navier–Stokes equations, *Phys. Fluids* 15 (2003) 3251–3254.
17. G. Ponce, R. Racke, T. Sideris, E. Titi, Global stability of large solutions to the 3d Navier–Stokes equations, *Comm. Math. Phys.* 159 (1994) 329–341.

INSTITUTO DE CIENCIAS MATEMÁTICAS, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS,
28049 MADRID, SPAIN

E-mail address: aenciso@icmat.es, renato.luca@icmat.es, dperalta@icmat.es